Consensus Pyramids

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Outline of talk

1. Recall basic definitions pertaining to the consensus of classification structures

2. Review some results for hierarchies and weak hierarchies

3. Consensus results for pyramids

1. Basic definitions and notation

A (simple) *hypergraph* on a finite set S is a set of non-empty subsets (the *clusters*) of S. Let $\mathcal{H}$ denote a set of hypergraphs on S such that $S \in H$ for every $H \in \mathcal{H}$.

A *consensus function* on $\mathcal{H}$ is a mapping $C : \mathcal{H}^k \to \mathcal{H}$ where k is a fixed positive integer. Elements of $\mathcal{H}^k$ are called *profiles* and are denoted by $\pi = (H_1, \ldots, H_k)$.

Our focus is on *counting rules* on $\mathcal{H}$, which are consensus functions C whereby a cluster A is placed in $C(\pi)$ if it satisfies criteria based on the number of times it appears among hypergraphs making up π.

A *hierarchy* is a hypergraph T with $\{x\} \in T$ for all $x \in S$, $S \in T$, and $A \cap B \in \{\emptyset, A, B\}$ for all $A, B \in T$. $\mathcal{T}$ denotes the set of all hierarchies on S.

A *weak hierarchy* (Bandelt and Dress) W on S is a hypergraph with $\{x\} \in W$ for all $x \in S$, $S \in W$ and $A \cap B \cap C \in \{A \cap B, A \cap C, B \cap C\}$ for all $A, B, C \in W$. $\mathcal{W}$ denotes the set of all weak hierarchies on S.

A *pyramid* (Diday) on S is a hypergraph P with $\{x\} \in P$ for all $x \in S$, $S \in P$, $A \cap B \in P \cup \{\emptyset\}$ for all $A, B \in P$, and there is a total ordering of S such that each cluster of P is an interval in this ordering. The set of all pyramids on S is denoted by $\mathcal{P}$.

It can be easily shown that $\mathcal{T} \subset \mathcal{P} \subset \mathcal{W}$.

2. Counting rules for $\mathcal{T}$ and $\mathcal{W}$

Let $A \subseteq S$ and $\pi = (H_1, ..., H_k) \in \mathcal{H}^k$. The *index* of A in π is

$$\gamma(A, \pi) = \frac{|\{i : A \in H_i\}|}{k}.$$

Counting rules can be described by a threshold t, where $M_t : \mathcal{H}^k \to \mathcal{H}$ is defined by

$$A \in M_t(\pi) \textit{ if and only if } \gamma(A, \pi) > t.$$

Domain and Range concerns for M_t

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Answer: $t = \frac{1}{2}$ and yields the *Majority Rule* (Margush & McMorris, 1981).

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Answer: $t = \frac{1}{2}$ and yields the *Majority Rule* (Margush & McMorris, 1981). (Abusing notation we allow $t = 1$ (really $t - \epsilon$) and have M_1 the *Unanimity Rule* commonly called the *strict consensus* in the biological literature.)

A new axiomatic characterization of the majority rule for hierarchies has been obtained. (McMorris & Powers, J. Classification submitted 2007)

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Answer: $t = \frac{2}{3}$. (Bandelt & Dress, 1989)

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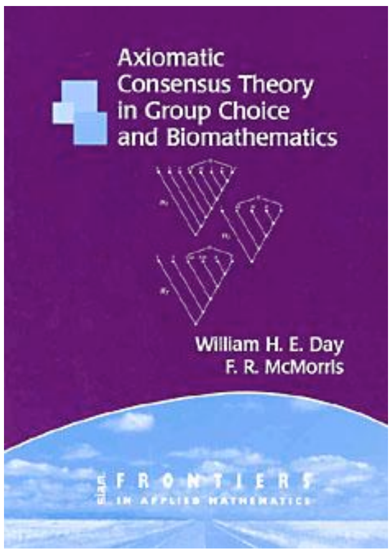
Answer: $t = \frac{1}{3}$. (Bandelt & Dress, 1989) In fact if π consists of m hierarchies and ℓ weak hierarchies ($m+\ell = k$), then $M_t(\pi) \in \mathcal{W}$ when $t = \frac{k+\ell}{3k}$. Axiomatic characterization of these consensus functions given by (McMorris & Powers, 1991) using the notion of "decisive families".

Along these lines I should mention the important, more general, work of Barthélemy, Leclerc, Monjardet, et al. in France, and Janowitz, et al. in the US.



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Answer: $t = 1$. (Lehel, McMorris & Powers, 1998)

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Answer: $t = 1$. (Lehel, McMorris & Powers, 1998)

All was not lost however, and other types of consensus rules for $\mathcal{P}$ were developed.

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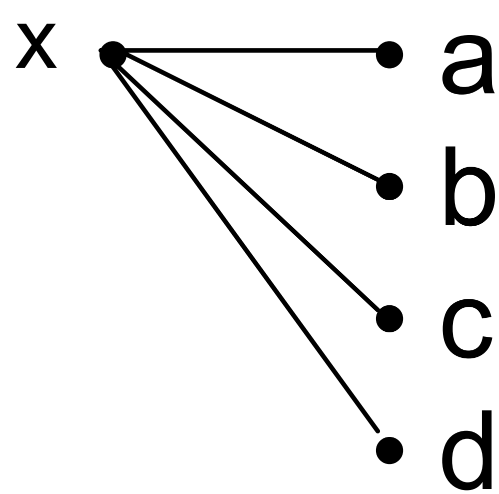
Motivated by talks at IFCS 2004 and IFCS 2006 by Diday where he introduced “spatial pyramids”: where the base interval is replaced by a type of grid-graph, and the clusters replaced by certain convex subgrids. The spatial pyramids can nicely be visualized, as has been shown by Diday.

In the Springer volume commemorating this Workshop, Powers and I take insight from previous work of ours (Lehel, McMorris & Powers, 1998)where we proposed the study of consensus of hypergraphs with the clusters taken as convex subsets (i.e.,subtrees) of a tree. We study the consensus of the simplest type of tree hypergraph (a when the tree is a star). Although general tree hypergraphs do not have the nice visualization properties of Diday's spatial pyramids, perhaps on "tree-like grids" a more spatial version may be possible. This is left for future investigation . . . .



A simple tree star.

A *star tree hypergraph* is a tree hypergraph where the underlying tree is a star graph (a graph with $n+1$ vertices, with n vertices of degree one and one vertex of degree n, the *central* vertex). Let $\mathcal{S}$ be the set of all star tree hypergraphs with vertex set S and $|S| \geq 3$.

We are concerned about $M_{\frac{1}{2}}(\pi)$ where $\pi \in \mathcal{S}^k$.

Let H_0 denote the hypergraph on S with no non-trivial clusters and for any $H \in \mathcal{S}$ with $H \neq H_0$ and $T \subseteq S$, let

$$T \cap H = T \cap A_1 \cap A_2 \cap \ldots \cap A_r$$

where $A_1, A_2, \ldots, A_r$ are the nontrivial clusters of H. For any nonempty subset $\mathcal{S}'$ of $\mathcal{S}$, let

$$c(\mathcal{S}') = min\{|T| : T \subset S \text{ and } T \cap H \neq \emptyset \; \forall H \in \mathcal{S}' \text{ with } H \neq H_0\}.$$

Result: For any nonempty subset $\mathcal{S}'$ of $\mathcal{S}$, if $c(\mathcal{S}') \leq 2$, then$M_{\frac{1}{2}}(\pi) \in \mathcal{S}$ for all $\pi \in (\mathcal{S}')^k$. Moreover, if $k \geq 3$, then there exists a subset $\mathcal{S}'$ of $\mathcal{S}$ such that $c(\mathcal{S}') = 3$ and $M_{\frac{1}{2}}(\pi) \notin \mathcal{S}$ for some $\pi \in (\mathcal{S}')^k$.

THE BEGINNING

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THANK YOU **EDWIN DIDAY** FOR YOUR WONDERFUL RESEARCH IDEAS OVER THE MANY YEARS!